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y-Wiener index of composite graphs

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ABSTRACT

Eliasi and Taeri [Extension of the Wiener index and Wiener polynomial, Appl. Math. Lett. 21 (2008) 916–921] introduced the notion of y-Wiener index of graphs as a generalization of the classical Wiener index and hyper Wiener index of graphs. They obtained some mathematical properties of this new defined topological index. In this paper, the join, Cartesian product, composition, disjunction and symmetric difference of graphs under y-Wiener index are computed. By these results most parts of a paper by Sagan et al. [The Wiener polynomial of a graph, Int. J. Quant. Chem. 60 (1996) 959–969] and another paper by Khalifeh et al. [The hyper-Wiener index of graph operations, Comput. Math. Appl. 56 (2008) 1402–1407] are generalized.

1. Introduction

Suppose G is a simple connected graph. As usual, the distance between the vertices u and v of G is denoted as $d_G(u, v)$ ($d(u, v)$ for short). It is defined as the length of a minimum path connecting them. The maximum of such numbers, $diam(G)$, is said to be the diameter of G. The number of pairs of vertices of G that are at distance k is denoted by $d(G, k)$. Notice that $d(G, 0)$ and $d(G, 1)$ represent the number of vertices and edges of G, respectively.

A topological index is a number related to a graph invariant under graph isomorphisms. Obviously, the number of vertices and edges of a given graph G are topological indices of G. The Wiener index [1] is the first distance based topological index defined by chemist Harold Wiener. This index is defined as the sum of all distances between vertices of G. After Wiener, too many authors continued the pioneering work of Wiener by introducing new topological indices. Nowadays, there are more than thousand topological indices and most of them have applications in chemistry, biochemistry, nanotechnology and computer science.

The Cartesian product $G \times H$ of graphs G and H is a graph such that $V(G \times H) = V(G) \times V(H)$, and any two vertices (a, b) and (u, v) are adjacent in $G \times H$ if and only if either $a = u$ and b is adjacent with v, or $b = v$ and a is adjacent with u, see [2, p. 185] for details. The join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2. The composition $G = G_1[G_2]$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph with vertex set $V_1 \times V_2$ and $u = (u_1, v_1)$ is adjacent with $v = (u_2, v_2)$ whenever (u_1 is adjacent with u_2) or ($u_1 = u_2$ and v_1 is adjacent with v_2). The disjunction $G \vee H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and (u_1, v_1) is adjacent to (u_2, v_2) whenever $u_1u_2 \in E(G)$ or $v_1v_2 \in E(H)$. The symmetric difference $G \oplus H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $E(G \oplus H) = \{(u_1, u_2)(v_1, v_2) \mid u_1v_1 \in E(G) \text{ or } u_2v_2 \in E(H) \text{ but not both}\}$ [3].

The Gamma function is a generalization of the well-known factorial function. It is defined as $\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt$. Remember that the Gamma function has the following properties: (i) $\Gamma(x + 1) = x\Gamma(x)$, and, (ii) if k is a non-negative

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integer then $\Gamma(k+1) = k!$. We now assume that y is a positive real number and G is a graph. Then $W(G, y)$, the y-Wiener index of G, is defined as $W(G, y) = \sum_{\{u,v\}\subseteq V(G)} \frac{\Gamma(d_G(u,v)+y)}{y\Gamma(d_G(u,v))}$. It is easy to see that this definition is equivalent to $W(G, y) = \sum_{t=1}^{l} \frac{\Gamma(t+y)}{y\Gamma(t)} d(G, t)$, where l denotes the diameter of G. One can see that $W(G, 1)$ is the classical Wiener index and $W(G, 2)$ is the well-known hyper-Wiener index of G [4].

Throughout this paper, C_n, P_n, K_n and S_n denote the cycle, path, complete and star graphs on n vertices. Also, $K_{m,n}$ denotes the complete bipartite graph. The complement of a graph G is a graph H on the same vertices such that two vertices of H are adjacent if and only if they are not adjacent in G. The graph H is usually denoted by $\overline{G}$. Our other notations are standard and taken mainly from [5,2].

2. Main results

Graovac and Pisanski were the first graph theorists to consider the problem of computing topological indices under graph operations. They computed an exact formula for the Wiener index of Cartesian product graphs [6]. In [7], Klavžar, Rajapakse and Gutman continued this problem by computing the Szeged index of Cartesian product graphs. In [8] Klavžar introduced the concept of PI partitions to find a formula for the PI index of Cartesian product of graphs. The present authors [9,10,6,11–19] continued this program to compute exact formulas for the hyper-Wiener, vertex PI, edge PI, the first Zagreb, the second Zagreb, the edge Wiener, the edge Szeged, the Wiener type indices of some graph operations. Zhang et al. [20] computed exact formulas for composite graphs under Kirschhoff index. There is also a polynomial approach for this problem in which a polynomial is related to a given topological index. By calculation of this polynomial and evaluating its derivative at $x = 1$, the topological index under consideration will be computed, see [21,22,3].

The aim of this section is to continue this program and compute the y-Wiener index of five graph operations, Cartesian product, composition, join, disjunction and symmetric difference. We begin with the following crucial lemma which will be used later.

Lemma 2.1. *Let G and H be graphs. Then we have:*

(a)

$$\begin{aligned}
&|V(G\times H)| = |V(G\vee H)| = |V(G[H])| = |V(G\oplus H)| = |V(G)|\cdot|V(H)|,\\
&|V(G+H)| = |V(G)| + |V(H)|,\\
&|E(G\times H)| = |E(G)|\cdot|V(H)| + |V(G)|\cdot|E(H)|,\\
&|E(G+H)| = |E(G)| + |E(H)| + |V(G)|\cdot|V(H)|,\\
&|E(G[H])| = |E(G)|\cdot|V(H)|^2 + |E(H)|\cdot|V(G)|,\\
&|E(G\vee H)| = |E(G)|\cdot|V(H)|^2 + |E(H)|\cdot|V(G)|^2 - 2|E(G)|\cdot|E(H)|,\\
&|E(G\oplus H)| = |E(G)|\cdot|V(H)|^2 + |E(H)|\cdot|V(G)|^2 - 4|E(G)|\cdot|E(H)|.
\end{aligned}$$

(b) *$G\times H$ is connected if and only if G and H are connected.*

(c) *If (a, c) and (b, d) are vertices of $G\times H$ then $d_{G\times H}((a, c), (b, d)) = d_G(a, b) + d_H(c, d)$.*

(d) *The Cartesian product, join, composition, disjunction and symmetric difference of graphs are associative and all of them are commutative except from composition.*

(e)

$$d_{G+H}(u, v) = \begin{cases} 0 & u = v \\ 1 & uv \in E(G) \text{ or } uv \in E(H) \text{ or } (u \in V(G)\ v \in V(H)) \\ 2 & otherwise. \end{cases}$$

(f)

$$d_{G[H]}((a, b), (c, d)) = \begin{cases} d_G(a, c) & a \neq c \\ 0 & a = c \,\&\, b = d \\ 1 & a = c \,\&\, bd \in E(H) \\ 2 & a = c \,\&\, bd \notin E(H). \end{cases}$$

(g)

$$d_{G\vee H}((a, b), (c, d)) = \begin{cases} 0 & a = c \,\&\, b = d \\ 1 & ac \in E(G) \text{ or } bd \in E(H) \\ 2 & otherwise. \end{cases}$$

(h)

$$d_{G\oplus H}((a, b), (c, d)) = \begin{cases} 0 & a = c \,\&\, b = d \\ 1 & ac \in E(G) \text{ or } bd \in E(H) \text{ but not both} \\ 2 & otherwise. \end{cases}$$

Proof. The parts (a)–(e) are consequences of definitions and some well-known results of the book of Imrich and Klavžar, [2]. For the proof of (f)–(h) we refer to [15]. □

Lemma 2.2. *The following statements hold:*

(1) $W(P_n, y) = \sum_{j=1}^{n-1} \frac{\Gamma(y)(n-j)}{(j-1)!} \prod_{i=1}^{j-1}(i+y)$,
(2) $W(K_n, y) = \binom{n}{2} \Gamma(y)$,
(3) $W(S_n, y) = \Gamma(y)(n-1) + (y+1)\Gamma(y)\binom{n-1}{2}$,
(4) $W(K_{m,n}, y) = \Gamma(y)mn + (y+1)\Gamma(y)\left(\binom{m}{2} + \binom{n}{2}\right)$,
(5) $W(C_n, y) = \begin{cases} \sum_{j=1}^{\frac{n}{2}-1} \frac{\Gamma(y)n}{(j-1)!} \prod_{i=1}^{j-1}(i+y) + \frac{n\Gamma\left(\frac{n}{2}+y\right)}{2y\Gamma\left(\frac{n}{2}\right)} & \text{if } n \text{ is even} \\ \sum_{j=1}^{\frac{n-1}{2}} \frac{\Gamma(y)n}{(j-1)!} \prod_{i=1}^{j-1}(i+y) & \text{if } n \text{ is odd.} \end{cases}$

Proposition 2.3. *Let G and H be connected graphs. Then*

$$W(G+H, y) = \Gamma(y)\left(\binom{|V(G)|}{2} + \binom{|V(H)|}{2} + |V(G)||V(H)|\right) + y\Gamma(y)(|E(\bar{G})| + |E(\bar{H})|).$$

Proof. By Lemma 2.1, we have:

$$\begin{aligned} W(G+H, y) &= \sum_{t=1}^{2} \frac{\Gamma(t+y)}{y\Gamma(t)} d(G+H, t) \\ &= \Gamma(y)d(G+H, 1) + (y+1)\Gamma(y)d(G+H, 2) \\ &= \Gamma(y)(|E(G)| + |E(H)| + |V(G)||V(H)|) \\ &\quad + (y+1)\Gamma(y)(|E(\bar{G})| + |E(\bar{H})|) \\ &= \Gamma(y)\left(\binom{|V(G)|}{2} + \binom{|V(H)|}{2} + |V(G)||V(H)|\right) + y\Gamma(y)(|E(\bar{G})| + |E(\bar{H})|), \end{aligned}$$

proving the result. □

Corollary 2.4. *Suppose $G_1, G_2, \ldots, G_n$ are graphs. Then*

$$W(G_1 + \cdots + G_n, y) = \Gamma(y)\left(\sum_{i=1}^{n} \binom{|V(G_i)|}{2} + \sum_{1 \le i < j \le n} |V(G_i)||V(G_j)|\right) + y\Gamma(y)\left(\sum_{i=1}^{n} |E(\bar{G}_i)|\right),$$

and $W(nG, y) = \Gamma(y)\left(n\binom{|V(G)|}{2} + \frac{n(n-1)}{2}|V(G)|^2\right) + y\Gamma(y)(n|E(\bar{G})|)$, where nG denotes the join of n copy of G.

Consider a complete n-partite graph $G = K_{m_1,m_2,\ldots,m_n}$ containing $v = |V(G)|$ vertices, Fig. 1. By definition, in this graph the set of vertices can be partitioned into subsets $V_1, V_2, \ldots, V_n$ of V such that for every i, $1 \le i \le n$, there is no edge between the vertices of V_i.

By the previous corollary, one can see that

$$W(K_{m_1,m_2,\ldots,m_n}, y) = \Gamma(y) \sum_{1 \le i < j \le n} m_i m_j + (1+y)\Gamma(y)\left(\sum_{i=1}^{n} \binom{m_i}{2}\right).$$

Proposition 2.5. *Let G and H be graphs. Then $W(G \vee H, y) = \Gamma(y)(|E(G)||V(H)|^2 + |E(H)||V(G)|^2 - 2|E(G)||E(H)|) + (y+1)\Gamma(y)(|V(G)||E(\bar{H})| + |V(H)||E(\bar{G})| + 2|E(\bar{G})||E(\bar{H})|)$.*

Proof. By Lemma 2.1 and the definition of disjunction,

$$\begin{aligned} W(G \vee H, y) &= \sum_{t=1}^{2} \frac{\Gamma(t+y)}{y\Gamma(t)} d(G \vee H, t) \\ &= \Gamma(y)d(G \vee H, 1) + (y+1)\Gamma(y)d(G \vee H, 2) \\ &= \Gamma(y)\left(|E(G)||V(H)|^2 + |E(H)||V(G)|^2 - 2|E(G)||E(H)|\right) \\ &\quad + (y+1)\Gamma(y)\left(|V(G)||E(\bar{H})| + |V(H)||E(\bar{G})| + 2|E(\bar{G})||E(\bar{H})|\right), \end{aligned}$$

proving the result. □

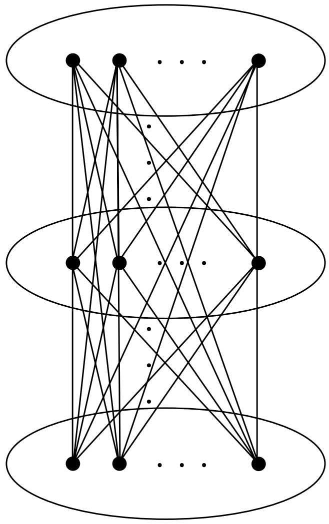

Fig. 1. The complete n-partite graph.

Proposition 2.6. *Let G and H be graphs. Then the y-Wiener of the symmetric difference of G and H is:* $W(G \oplus H, y) = \Gamma(y)(|E(G)||V(H)|^2 + |E(H)||V(G)|^2 - 4|E(G)||E(H)|) + (y+1)\Gamma(y)(2|E(G)||E(H)| + |V(H)||E(\bar{G})| + |V(G)||E(\bar{H})| + 2|E(\bar{H})||E(\bar{G})|)$.

Proof. By Lemma 2.1 and the definition of symmetric difference of two graphs, we have:

$$\begin{aligned} W(G \oplus H, y) &= \sum_{t=1}^{2} \frac{\Gamma(t+y)}{y\Gamma(t)} d(G \oplus H, t) \\ &= \Gamma(y) d(G \oplus H, 1) + (y+1)\Gamma(y) d(G \oplus H, 2) \\ &= \Gamma(y)\left(|E(G)||V(H)|^2 + |E(H)||V(G)|^2 - 4|E(G)||E(H)|\right) \\ &\quad + (y+1)\Gamma(y)\left(2|E(G)||E(H)| + |V(H)||E(\bar{G})| + |V(G)||E(\bar{H})| \right.\\ &\quad \left. + \ 2|E(\bar{H})||E(\bar{G})|\right), \end{aligned}$$

proving the result. □

Proposition 2.7. *Let G and H be graphs. Then* $W(G[H], y) = \Gamma(y)|V(G)|\binom{|V(H)|}{2} + y\Gamma(y)|V(G)||E(\bar{H})| + |V(H)|^2 W(G, y)$.

Proof. Let $d_1(G[H], k)$ and $d_2(G[H], k)$ denote the number of 2-subsets $\{(a, b), (x, y)\}$ such that $[d_{G[H]}((a, b), (x, y)) = k, a = x]$ and $[d_{G[H]}((a, b), (x, y)) = k, a \neq x]$, respectively. In the first case, by Lemma 2.1 $d_{G[H]}((a, b), (x, y)) \leq 2$ and so,

$$\begin{aligned} W(G[H], y) &= \sum_{t=1}^{2} \frac{\Gamma(t+y)}{y\Gamma(t)} d_1(G[H], t) + \sum_{t=1}^{d_G} \frac{\Gamma(t+y)}{y\Gamma(t)} d_2(G[H], t) \\ &= \Gamma(y) d_1(G[H], 1) + (y+1)\Gamma(y) d_1(G[H], 2) + |V(H)|^2 W(G, y) \\ &= \Gamma(y)|V(G)||E(H)| + (y+1)\Gamma(y)|V(G)||E(\bar{H})| + |V(H)|^2 W(G, y) \\ &= \Gamma(y)|V(G)|\binom{|V(H)|}{2} + y\Gamma(y)|V(G)||E(\bar{H})| + |V(H)|^2 W(G, y), \end{aligned}$$

proving the result. □

Proposition 2.8. *Let G and H be graphs. Then* $W(G \times H, y) = |V(H)|W(G, y) + |V(G)|W(H, y) + 2\sum_{t=1}^{l} \frac{\Gamma(t+y)}{y\Gamma(t)} \sum_{i=1}^{t-1} d(G, i) d(H, t-i)$, *where* $l = diam(G) + diam(H)$.

Proof. By Lemma 2.1(c) and an inductive argument, one can prove $d(G \times H, t) = 2\sum_{i=1}^{t-1} d(G, i) d(H, t-i) + |V(H)| d(G, t) + |V(G)| d(H, t)$. Therefore,

$$W(G \times H, y) = \sum_{t=1}^{l} \frac{\Gamma(t+y)}{y\Gamma(t)} d(G \times H, t)$$

$$= \sum_{t=1}^{l} \frac{\Gamma(t+y)}{y\Gamma(t)} \left(2\sum_{i=1}^{t-1} d(G,i)d(H,t-i) + |V(H)|d(G,t) + |V(G)|d(H,t) \right)$$
$$= |V(H)|W(G,y) + |V(G)|W(H,y) + 2\sum_{t=1}^{l} \frac{\Gamma(t+y)}{y\Gamma(t)} \sum_{i=1}^{t-1} d(G,i)d(H,t-i),$$

which completes our result. □

Proposition 2.9. *Let G and H be graphs. Then*

$$W(G\times H,y) \le \frac{1}{y}\binom{|V(G\times H)|}{2}\prod_{i=0}^{y-1}(diam(G)+diam(H)+i),$$

with equality if and only if both of G and H are complete graphs.

Proof. By Lemma 2.1(c) and definition,

$$\begin{aligned} W(G\times H,y) &= \sum_{\{(a,x),(b,z)\}} \frac{\Gamma(d_{G\times H}((a,x),(b,z))+y)}{y\Gamma(d_{G\times H}((a,x),(b,z)))} \\ &= \sum_{\{(a,x),(b,z)\}} \frac{\Gamma(d_G(a,b)+d_H(x,z)+y)}{y\Gamma(d_G(a,b)+d_H(x,z))} \\ &= \frac{1}{y}\sum_{\{(a,x),(b,z)\}} \prod_{i=0}^{y-1}(d_G(a,b)+d_H(x,z)+i) \\ &\le \frac{1}{y}\sum_{\{(a,x),(b,z)\}} \prod_{i=0}^{y-1}(diam(G)+diam(H)+i) \\ &= \frac{1}{y}\binom{|V(G\times H)|}{2}\prod_{i=0}^{y-1}(diam(G)+diam(H)+i). \end{aligned}$$

We notice that for all pairs $(a,x),(b,z) \in V(G\times H)$, $d(a,b) \le diam(G)$ and $d(x,z) \le diam(H)$. So, the equality holds if and only if $diam(H) = diam(G) = 1$. □

3. Concluding remarks

In this paper exact formulas for a y-Wiener index of some graph operations are obtained. By the results given here, it is possible to improve most parts of a paper by Sagan et al. [3] and another paper by Khalifeh et al. [15] generalized to a y-Wiener index. We believed our proofs are simple and better than those given in the mentioned papers.

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